The propagation of long waves into a semi-infinite channel in a rotating system

By J. CREASE
National Institute of Oceanography, Wormley, Surrey

(*Received* 23 *December* 1957)

SUMMARY

In this paper the propagation of long waves (tides) into a canal is studied. The canal is assumed to be rotating at constant angular velocity and the depth of the fluid is uniform.

The rate of rotation can have a considerable effect on the amplitude of the unattenuated modes in the channel. This is due partly to the modification of the known solution when the rotation is zero, and partly to the fact that even if there is only a single semi-infinite barrier unattenuated waves of a special type (Kelvin waves) may be propagated in the rotating system into what is normally called the 'shadow' region behind the barrier (Crease 1956).

The purpose of this theoretical investigation is to seek a partial explanation of the behaviour of tides and storm surges in the North Sea. For this reason two models of this region are discussed. In Example I the model is of two parallel semi-infinite barriers in the path of a plane progressive wave, and in Example II there are two parallel barriers, one semi-infinite and the other infinite.

The ratio of the observed height of the semi-diurnal M_2 tide in the North Sea to the height of the incident tide from the Atlantic lies between the results predicted by the two models and is in closer agreement with the model of Example I.

1. INTRODUCTION

It has been shown in a previous paper (Crease 1956) that long waves incident normally on a semi-infinite barrier in a rotating system give rise to waves of Kelvin type which are propagated without attenuation into the region behind the barrier. The direction of propagation is parallel to the barrier and the crest heights decrease exponentially in the direction normal to the barrier. When the ratio of the wave frequency σ to the angular velocity ω of the rotating system is such that $2\omega/\sigma > 3/5$, the amplitude of the Kelvin wave at the barrier is greater than that of the incident waves.

This effect appears to explain qualitatively the formation of the semi-diurnal tides in the North Sea; for, to a first approximation, the British Isles may be regarded as a semi-infinite barrier across the path of the M_2 tide propagated in from the Atlantic.

In this paper the effect of the continental coast on the formation of the Kelvin wave is examined. Two extreme cases are considered as an approximation to the continental boundary. In Example I (§ 3 to § 5) the boundary is taken to be a second semi-infinite barrier parallel to the first and a distance $2a$ behind it. In Example II (§ 6 to § 11) it is assumed to be an infinite barrier distant $2a$ behind the semi-infinite barrier.

In neither of these systems is there a barrier corresponding to the southern end of the North Sea. Taylor (1921) has considered the reflection of Kelvin waves at the end of a long gulf, but here we are only concerned with the transmission of waves into a channel from outside. The possibility of resonance effects is thus ruled out.

The problem when $\omega = 0$ is a familiar one in acoustics and scalar electromagnetic theory, and Heins (1948) has solved the problem of an incident electric field parallel to the edges of the barriers. It is his method of solution, based on the Wiener–Hopf method, that is used in this paper. Vajnshtejn (1948) has also solved this problem and the acoustic problem, and investigates his solutions in more detail.

In the case of a single barrier, rotation gives rise to the Kelvin wave mentioned above in addition to the usual diffracted wave diverging from the edge of the barrier. When there are two semi-infinite barriers the parts of the diffracted waves propagating into the channel may be regarded loosely as giving rise to the 'channel' modes of acoustics ($\omega = 0$) which, by the rotation of the system, become waves of Kelvin and Poincaré type (Proudman 1953, p. 265). In addition, the Kelvin waves arising from the barriers individually also propagate down the channel. The amplitude of the waves in the channel will depend on its width, which in this paper is restricted to be less than half a wavelength. This condition is satisfied for the North Sea. The observational data for the North Sea lie between the two cases, and is in better agreement with the model proposed as Example I.

2. The differential equation for the wave height ζ

The equations governing the propagation of long waves in a system rotating at angular velocity ω are, subject to certain approximations (cf. Lamb 1932, p. 318),

$$\left.\begin{aligned} \frac{\partial u}{\partial t} - fv &= -g\frac{\partial \zeta}{\partial x}, \\ \frac{\partial v}{\partial t} + fu &= -g\frac{\partial \zeta}{\partial y}, \\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= -\frac{1}{h}\frac{\partial \zeta}{\partial t}, \end{aligned}\right\} \tag{1}$$

where u, v are the horizontal components of particle velocity parallel to x, y, h is the depth of water, and $f = 2\omega$. From equations (1) the equation for ζ

is found to be

$$\frac{\partial^2\zeta}{\partial x^2}+\frac{\partial^2\zeta}{\partial y^2}=\frac{1}{gh}\left(\frac{\partial^2\zeta}{\partial t^2}+f^2\zeta\right). \tag{2}$$

The boundary condition at a barrier is that the normal component of velocity is zero. When the barrier is along the x-axis, we have $v = 0$ and, in terms of ζ,

$$\frac{\partial^2\zeta}{\partial y\partial t}=f\frac{\partial\zeta}{\partial x} \tag{3}$$

on the barrier.

If the motion is periodic with time, we may replace $\zeta(x,y,t)$ by $\zeta(x,y)\exp(-i\sigma t)$, say. Equations (2) and (3) then become

$$\frac{\partial^2\zeta}{\partial x^2}+\frac{\partial^2\zeta}{\partial y^2}+k^2\zeta=0, \tag{4}$$

where $k^2=(\sigma^2-f^2)/gh$, and

$$\frac{\partial\zeta}{\partial y}=\frac{if}{\sigma}\frac{\partial\zeta}{\partial x} \tag{5}$$

on the barrier. It will be convenient to let $if/\sigma = \tan i\beta$, and to assume that k has a small positive imaginary part which may be taken to be zero when the solution of the problem has been obtained.

EXAMPLE I. TWO SEMI-INFINITE BARRIERS

3. The Integral Equations

Let the barriers be at $y = \pm a$, $x > 0$ (figure 1). By Green's theorem $\zeta(x,y)$ may be expressed as a contour integral round a contour Γ surrounding the point (x,y). Thus,

$$\zeta(x,y)=\int_\Gamma\left[G(x,y;x_0,y_0)\frac{\partial}{\partial n_0}\zeta(x_0,y_0)-\zeta(x_0,y_0)\frac{\partial}{\partial n_0}G(x,y;x_0,y_0)\right]ds_0,$$

where $\partial/\partial n_0$ is differentiation along the outward normal to Γ, and ds_0 is an element of Γ, which is taken to be a circle indented round the barriers and whose radius tends to infinity (figure 1).

G is the free-space Green's function given by (Morse & Feshbach 1953, p. 811)

$$G(x,y;x_0,y_0)=\tfrac{1}{4}iH_0^{(1)}[k\{(x-x_0)^2+(y-y_0)^2\}^{1/2}], \tag{6}$$

where $H_0^{(1)}$ is the usual notation for a Bessel function of the third kind.

By using the boundary condition (5) and integrating by parts (Crease 1956), it is found that

$$\zeta(x,y)=\frac{1}{\cos i\beta}\int_0^\infty\left[\left\{[\zeta]\left(\cos i\beta\frac{\partial}{\partial y_0}+\sin i\beta\frac{\partial}{\partial x_0}\right)G\right\}_{y_0=-a}+\right.$$
$$\left.+\left\{[\zeta]\left(\cos i\beta\frac{\partial}{\partial y_0}+\sin i\beta\frac{\partial}{\partial x_0}\right)G\right\}_{y_0=a}\right]dx_0+\zeta_i, \tag{7}$$

where $[\zeta]$ is the limit as $\delta\to 0$ of $\zeta(x_0,y_0+\delta)-\zeta(x_0,y_0-\delta)$, and it is assumed that $[\zeta]$ is bounded at infinity. Here ζ_i is the prescribed incident wave and is taken to be a plane wave at an angle θ_0 to the barrier. Thus,

$$\zeta_i = \exp ik(x\cos\theta_0 + y\sin\theta_0).$$

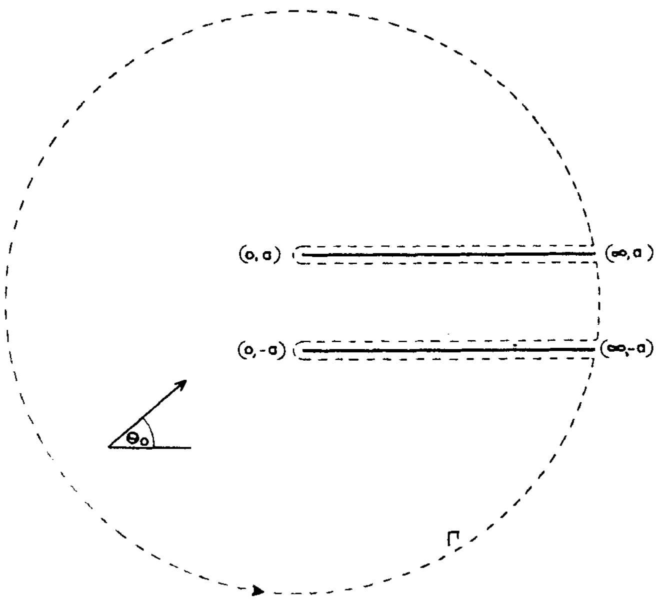

Figure 1.

The boundary condition is that

$$\left(\cos i\beta\,\frac{\partial}{\partial y} - \sin i\beta\,\frac{\partial}{\partial x}\right)\zeta$$

shall vanish on $y = \pm a$, $x > 0$. Thus, by operating on (7) with

$$\cos i\beta\left(\cos i\beta\,\frac{\partial}{\partial y} - \sin i\beta\,\frac{\partial}{\partial x}\right)$$

and setting $y = \pm a$, $x > 0$, we obtain the following pair of integral equations of Wiener–Hopf type for $[\zeta(x, a)]$ and $[\zeta(x, -a)]$. When these are solved, (7) gives $\zeta(x, y)$. Thus, on $y = \mp a$,

$$g_{1,2}(x) = q_{1,2}(x) + \left(\cos i\beta\,\frac{\partial}{\partial y} - \sin i\beta\,\frac{\partial}{\partial x}\right)\times$$
$$\times\int_{-\infty}^{\infty}\left\{\left[f_1(x_0)\left(\cos i\beta\,\frac{\partial}{\partial y_0} + \sin i\beta\,\frac{\partial}{\partial x_0}\right)G(x, y; x_0, y_0)\right]_{y_0=-a} + \right.$$
$$\left. + \left[f_2(x_0)\left(\cos i\beta\,\frac{\partial}{\partial y_0} + \sin i\beta\,\frac{\partial}{\partial x_0}\right)G(x, y; x_0, y_0)\right]_{y_0=a}\right\} dx_0, \quad (8)$$

where

$$\left.\begin{aligned} f_{1,2}(x) &= [\zeta(x, \mp a)] && x > 0,\\ &= 0 && x < 0,\\ q_{1,2}(x) &= ik\cos i\beta\sin(\theta_0 - i\beta)\exp ik(x\cos\theta_0 \mp a\sin\theta_0) && x > 0,\\ &= 0 && x < 0,\\ g_{1,2}(x) &= 0 && x > 0, \end{aligned}\right\} \tag{9}$$

and $g_{1,2}(x)$ are defined by equations (12) for $x < 0$. These and following pairs of equations have been written in a condensed form. The upper and lower signs in the equations refer to the first and second suffices respectively (e.g. $f_{1,2}(x) = [\zeta(x, \mp a)]$ implies the two equations

$$f_1(x) = [\zeta(x, -a)], \qquad f_2(x) = [\zeta(x, a)]).$$

Equations (8) may be added and subtracted; and since

$$G(x, -a; x_0, -a) = G(x, a; x_0, a)$$

and $$G(x, -a; x_0, a) = G(x, a; x_0, -a) = G(x, 0; x_0, -2a),$$

we have

$$g_{3,4}(x) = q_{3,4}(x) + \left(\cos i\beta\frac{\partial}{\partial y} - \sin i\beta\frac{\partial}{\partial x}\right) \times \\ \times \int_{-\infty}^{\infty} f_{3,4}(x)\left(\cos i\beta\frac{\partial}{\partial y_0} + \sin i\beta\frac{\partial}{\partial x_0}\right)(G_{y_0 = -2a} + G_{y_0 = 0})\,dx_0 \tag{10}$$

on $y = 0$, where $g_{3,4} = g_1 \pm g_2$ etc. We may now take Fourier transforms of these equations, denoting transforms by corresponding capital letters, and obtain

$$G_{3,4}(\alpha) = Q_{3,4}(\alpha) + F_{3,4}(\alpha)\left(\cos i\beta\frac{\partial}{\partial y} - i\alpha\sin i\beta\right)\left(\cos i\beta\frac{\partial}{\partial y_0} - i\alpha\sin i\beta\right) \times \\ \times \{K(\alpha, y, y_0)_{y = y_0 = 0} \pm K(\alpha, y, y_0)_{y = 0, y_0 = -2a}\}, \tag{11}$$

where $K(\alpha, y, y_0)$ is the transform of $G(x, y; 0, y_0)$ and is given (Heins 1948) by

$$K(\alpha, y, y_0) = \tfrac{1}{2}i\exp[i|y - y_0|(k^2 - \alpha^2)^{1/2}]/(k^2 - \alpha^2)^{1/2}. \tag{12}$$

Equations (11) may be conveniently written as

$$G_{3,4}(\alpha) = Q_{3,4}(\alpha) - F_{3,4}(\alpha)K_{3,4}(\alpha), \tag{13}$$

in which $K_{3,4}(\alpha)$ are explicitly

$$K_{3,4}(\alpha) = (\alpha^2 - k^2\cos^2 i\beta)(k^2 - \alpha^2)^{1/2}\exp ia(k^2 - \alpha^2)^{1/2}\binom{i\cos}{\sin}a(k^2 - \alpha^2)^{1/2}. \tag{14}$$

Also, from (9),

$$Q_{3,4}(\alpha) = 2k(\alpha - k\cos\theta_0)^{-1}\cos i\beta\sin(\theta_0 - i\beta)\binom{\cos}{-i\sin}(ak\sin\theta_0). \tag{15}$$

These equations are determinate, as all the transforms have a common strip of regularity $-\mathscr{I}\{k\} < \mathscr{I}\{\alpha\} < 0$ (or $< \mathscr{I}\{k\cos\theta_0\}$ if this is negative). The argument is the same as for the case of a single barrier (Crease 1956). The transforms $K_{3,4}(\alpha)$ may be decomposed into the form $K_{3,4}(\alpha) = K^-_{3,4}(\alpha)/K^+_{3,4}(\alpha)$, where $K^-_{3,4}(\alpha)$ are regular and free of zeros for $\mathscr{I}\{\alpha\} < \mathscr{I}\{k\}$, and $K^+_{3,4}(\alpha)$ are regular and free of zeros for $\mathscr{I}\{\alpha\} > -\mathscr{I}\{k\}$.

Equations (13) may now be written as

$$G_{3,4}(\alpha)K^+_{3,4}(\alpha) - Q_{3,4}(\alpha)\{K^+_{3,4}(\alpha) - K^+_{3,4}(k\cos\theta_0)\}$$
$$= Q_{3,4}(\alpha)K^+_{3,4}(k\cos\theta_0) - F_{3,4}(\alpha)K^-_{3,4}(\alpha). \quad (16)$$

In this form the left-hand side of these equations are regular for $\mathscr{I}\{\alpha\} > \mathscr{I}\{k\}$, and the right-hand side for $\mathscr{I}\{\alpha\} < 0$ (or $< \mathscr{I}\{k\cos\theta_0\} < 0$). Thus each side of both equations (16) is an analytical continuation in their (overlapping) half-planes of entire functions $E_{3,4}(\alpha)$ respectively. Now $K^+_{3,4}(\alpha)$ are $O(\alpha^{-1/2})$ and $K^-_{3,4}(\alpha)$ are $O(\alpha^{1/2})$ as $|\alpha| \to \infty$ (see below), and it follows from the known behaviour of the other functions that the left-hand side is $O(\alpha^{-1/2})$ and the right-hand side is $O(\alpha^{1/2})$. By an extension to Liouville's theorem (Titchmarsh 1939, p. 85), the entire functions must be constants; and, from the behaviour of the left-hand side as $|\alpha| \to \infty$, the constants are zero. Therefore it follows from equations (16) that

$$F_{3,4}(\alpha) = Q_{3,4}(\alpha)K^+_{3,4}(k\cos\theta_0)/K^-_{3,4}(\alpha). \quad (17)$$

It remains to determine the factorization of equations (14) explicitly. From (14),

$$K_3(\alpha) = \frac{K_3^-(\alpha)}{K_3^+(\alpha)} = \frac{(\alpha^2 - k^2\cos^2 i\beta)}{(k^2-\alpha^2)^{1/2}}\, i\exp\left[\frac{2ia}{\pi}(k^2-\alpha^2)^{1/2} \times \right.$$
$$\left. \times \left\{\tan^{-1}\left(\frac{k+\alpha}{k-\alpha}\right)^{1/2} + \tan^{-1}\left(\frac{k-\alpha}{k+\alpha}\right)^{1/2}\right\}\right] \times$$
$$\times \prod_{n=1}^{\infty}\left[1 - \frac{4a^2(k^2-\alpha^2)}{(2n-1)^2\pi^2}\right].$$

Thus

$$K_3^-(\alpha) = i\frac{\alpha - k\cos i\beta}{(k-\alpha)^{1/2}}\exp\left[\frac{2ia}{\pi}(k^2-\alpha^2)^{1/2}\tan^{-1}\left(\frac{k+\alpha}{k-\alpha}\right)^{1/2} + \chi_0(\alpha)\right] \times$$
$$\times \prod_{n=1}^{\infty}\left[\left(1 - \frac{4a^2k^2}{(2n-1)^2\pi^2}\right)^{1/2} + \frac{2ia\alpha}{(2n-1)\pi}\right]\exp\left[-\frac{2ia\alpha}{(2n-1)\pi}\right], \quad (18)$$

$$\frac{1}{K_3^+(\alpha)} = \frac{\alpha + k\cos i\beta}{(k+\alpha)^{1/2}}\exp\left[\frac{2ia}{\pi}(k^2-\alpha^2)^{1/2}\tan^{-1}\left(\frac{k-\alpha}{k+\alpha}\right)^{1/2} - \chi_0(\alpha)\right] \times$$
$$\times \prod_{n=1}^{\infty}\left[\left(1 - \frac{4a^2k^2}{(2n-1)^2\pi^2}\right)^{1/2} - \frac{2ia\alpha}{(2n-1)\pi}\right]\exp\left[\frac{2ia\alpha}{(2n-1)\pi}\right], \quad (19)$$

where $\chi_0(\alpha)$ is an entire function introduced to make $K_3^-(\alpha)$ and $K_3^+(\alpha)$ algebraic as $|\alpha| \to \infty$. Asymptotically we find that as $|\alpha| \to \infty$, $\mathscr{I}\{\alpha\} < \mathscr{I}\{k\}$ (cf. Heins 1948)

$$K_3^-(\alpha) \sim -(\alpha/2)^{1/2}\exp\left\{\frac{ia\alpha}{\pi}\left[1 - \gamma + \log\frac{i\pi}{2ak}\right] + \chi_0(\alpha)\right\},$$

where γ is Euler's constant.

Therefore, if

$$\chi_0(\alpha) = -\frac{ia\alpha}{\pi}\left[1-\gamma+\log\frac{i\pi}{2ak}\right], \tag{20}$$

then $K_3^-(\alpha) \sim -(\alpha/2)^{1/2}$ as $|\alpha| \to \infty$, $\mathscr{I}\{\alpha\} < \mathscr{I}\{k\}$; and similarly $K_3^+(\alpha) \sim (2/\alpha)^{1/2}$ as $|\alpha| \to \infty$, $\mathscr{I}\{\alpha\} > -\mathscr{I}\{k\}$. Similarly, for $K_4(\alpha)$,

$$K_4^-(\alpha) = a(\alpha - k\cos i\beta)\exp\left[\frac{2ia}{\pi}(k^2-\alpha^2)^{1/2}\tan^{-1}\left(\frac{k+\alpha}{k-\alpha}\right)^{1/2} + \chi_1(\alpha)\right]\times$$
$$\times\prod_{n=1}^{\infty}\left[\left(1-\frac{a^2k^2}{n^2\pi^2}\right)^{1/2} + \frac{ia\alpha}{n\pi}\right]\exp\left(-\frac{ia\alpha}{n\pi}\right), \tag{21}$$

$$\frac{1}{K_4^+(\alpha)} = (\alpha + k\cos\beta)\exp\left[\frac{2ia}{\pi}(k^2-\alpha^2)^{1/2}\tan^{-1}\left(\frac{k-\alpha}{k+\alpha}\right)^{1/2} - \chi_1(\alpha)\right]\times$$
$$\times\prod_{n=1}^{\infty}\left[\left(1-\frac{a^2k^2}{n^2\pi^2}\right)^{1/2} - \frac{ia\alpha}{n\pi}\right]\exp\left(\frac{ia\alpha}{n\pi}\right), \tag{22}$$

with

$$\chi_1(\alpha) = -\frac{ia\alpha}{\pi}\left[1-\gamma+\log\frac{2i\pi}{ak}\right]. \tag{23}$$

The asymptotic behaviour of $K_4^-(\alpha)$ and $K_4^+(\alpha)$ is then

$$K_4^-(\alpha) \sim \tfrac{1}{2}a(ia\alpha/\pi)^{1/2}, \qquad K_4^+(\alpha) \sim (-\pi/ia\alpha)^{1/2}.$$

The solution of the integral equations is now given by the Fourier transforms of equations (17), but these transforms will not be needed explicitly as ζ may be expressed directly in terms of $F_{3,4}(\alpha)$.

4. The wave height ζ

The integrals in (6) are convolutions and may be rewritten as

$$\zeta(x,y) = \frac{1}{2\pi\cos i\beta}\int_{-\infty-i\epsilon}^{\infty-i\epsilon}\left[F_1(\alpha)\left\{\left(\cos i\beta\frac{\partial}{\partial y_0} - i\alpha\sin i\beta\right)K(\alpha,y,y_0)\right\}_{y_0=-a} + \right.$$
$$\left. + F_2(\alpha)\left\{\left(\cos i\beta\frac{\partial}{\partial y_0} - i\alpha\sin i\beta\right)K(\alpha,y,y_0)\right\}_{y_0=a}\right]\exp(i\alpha x)\,d\alpha +$$
$$+\exp ik(x\cos\theta_0 + y\sin\theta_0) \tag{24}$$

where $-\mathscr{I}\{k\} < \epsilon < 0$ (or $< \mathscr{I}\{k\cos\theta_0\} < 0$). Particular interest lies in the wave amplitude at large distances from the edges of the barriers. This is given by the residues at the poles of the integrand; the branch line integrals obtained by deforming the contour only give a contribution which tends to zero at large distances from the edges. The poles are at $\alpha = k\cos\theta_0$, $k\cos i\beta$, and the zeros of the infinite products of (18) and (21). These latter zeros give rise to terms decaying exponentially to zero in the x-direction, provided $0 < 2ak < \pi$.

We consider first the pole at $\alpha = k\cos\theta_0$. If $x < 0$, the contour of equation (24) may be closed below, and there is no contribution of the pole to the integral. If $x > 0$, the residue is found to be

$$-\exp ik(x\cos\theta_0 + y\sin\theta_0) \quad \text{for } y > -a,$$

and

$$\frac{\sin(\theta_0 - i\beta)}{\sin(\theta_0 + i\beta)}\exp ik\{x\cos\theta_0 - (y+2a)\sin\theta_0\} \tag{25}$$

for $y < -a$. Thus, these terms together with the incident wave represent the reflection pattern obtained by a geometrical construction (apart from a phase change in the reflected wave due to rotational effects).

Next we consider the residue at the pole $\alpha = k\cos i\beta$. There is again no contribution for $x < 0$. For $x > 0$, $y > a$, the residue is found to be

$$\frac{4\cos i\beta}{\sin(\theta_0+i\beta)}\times$$
$$\times\exp\left[\frac{iak}{\pi}\left\{-\theta_0\sin\theta_0+i\beta\sin i\beta+(\cos i\beta-\cos\theta_0)\left(1-\gamma+\log\frac{i\pi}{2ak}\right)\right\}\right]\times$$
$$\times[\cos\tfrac{1}{2}\theta_0\sin\tfrac{1}{2}i\beta\cos^{1/2}(ak\sin\theta_0)\cos^{1/2}(ak\sin i\beta)\exp i\psi_0+$$
$$+\tfrac{1}{2}i\{\sin\theta_0\sin i\beta\sin(ak\sin\theta_0)\sin(ak\sin i\beta)\}^{1/2}\exp i\psi_1]\times$$
$$\times\exp ik\{x\cos i\beta+(y-a)\sin i\beta\}, \qquad (26)$$

where ψ_0, ψ_1 are real phase factors which tend to zero when $ak\to 0$. When $a > y > -a$, the residue ζ_c at $\alpha = k\cos i\beta$ is

$$\zeta_c = \frac{2\cos i\beta}{\sin(\theta_0+i\beta)}\exp\left[\frac{iak}{\pi}\left\{-\theta_0\sin\theta_0-(\pi-i\beta)\sin i\beta-\right.\right.$$
$$\left.\left.-(\cos\theta_0-\cos i\beta)\left(1-\gamma+\log\frac{i\pi}{2ak}\right)\right\}\right]\left[\cos\tfrac{1}{2}\theta_0\sin\tfrac{1}{2}i\beta\left\{\frac{\cos(ak\sin\theta_0)}{\cos(ak\sin i\beta)}\right\}^{1/2}\times\right.$$
$$\left.\times\exp i\psi_0+\frac{1}{2}\left\{\sin\theta_0\sin i\beta\,\frac{\sin(ak\sin\theta_0)}{\sin(ak\sin i\beta)}\right\}^{1/2}\exp i\psi_1\right]\times$$
$$\times\exp ik\{x\cos i\beta+(y+a)\sin i\beta\}. \qquad (27)$$

Finally for $y < -a$ there is no pole at $\alpha = k\cos i\beta$. Thus the wave height at large distances from the barrier (and from the edge of the geometrical shadow) is determined by the incident wave and equations (25), (26) and (27).

5. Discussion

We may consider first the form of the solution when the barriers are close together ($ak\to 0$). Then, for $y > a$, the expression (26) becomes

$$\frac{4\cos i\beta}{\sin(\theta_0+i\beta)}\cos\tfrac{1}{2}\theta_0\cos\tfrac{1}{2}i\beta\exp ik\{x\cos i\beta+(y-a)\sin i\beta\}, \qquad (28)$$

and equation (28) becomes

$$\zeta_c = \frac{2\cos i\beta\cos\tfrac{1}{2}\theta_0}{\sin(\theta_0+i\beta)}(\sin\tfrac{1}{2}i\beta+\sin\tfrac{1}{2}\theta_0)\exp ik\{x\cos i\beta+(y+a)\sin i\beta\}. \qquad (29)$$

As is to be expected, (28) is just the solution for the wave height in the shadow region behind a single semi-infinite barrier (cf. the particular case of normal incidence discussed by Crease (1956)).

The second term of (29) shows perhaps rather more clearly than (27) the effect of rotation in modifying the 'channel' mode which exists when $\omega = 0$ (see below), whilst the first term gives the additional Kelvin wave which arises purely through rotation of the system whether there be one barrier or two.

When the rate of rotation becomes small ($\beta \to 0$), the residue given by (26) approaches zero whilst (27) becomes

$$\zeta_c = \exp\left[\frac{iak}{\pi}\left\{-\theta_0 \sin\theta_0 + (1-\cos\theta_0)\left(1-\gamma+\log\frac{i\pi}{2ak}\right)\right\}\right]\times$$
$$\times\left[\frac{(\sin ak\sin\theta_0)}{ak\sin\theta_0}\right]^{1/2}\exp i\psi_1 \exp ikx, \quad (30)$$

with $\beta = 0$ in ψ_1. This result corresponds to the usual channel mode without rotation, and has its counterpart in the acoustic and electromagnetic problem.

Figure 2 shows for comparison the amplitude of the waves transmitted into the channel when $\cosh\beta = 1$ $(\omega = 0)$ and when $\cosh\beta = 2$ $(\omega/\sigma = \frac{1}{4}\sqrt{3})$ for three different widths of the channel. The latter value

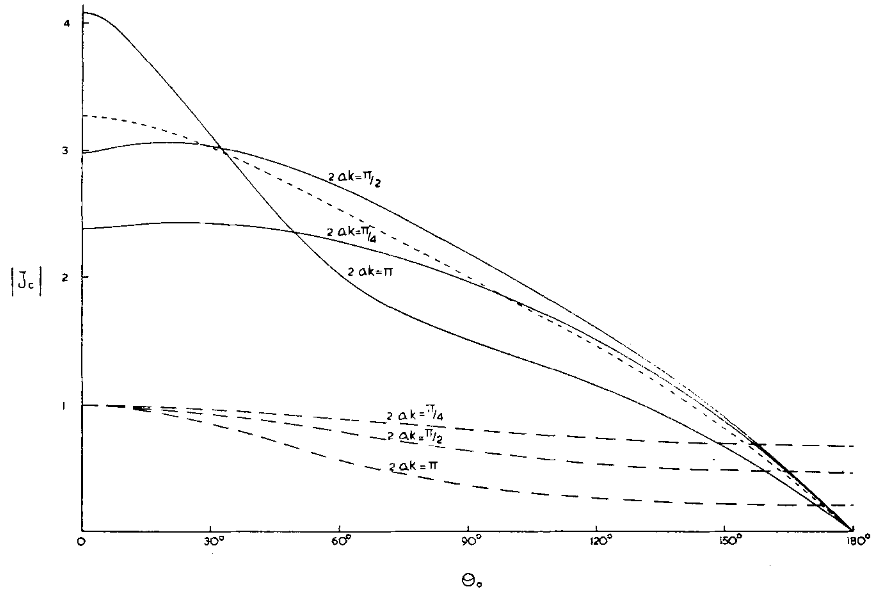

Figure 2. Amplitude of ζ_c on $y = -a$ for $\cosh\beta = 2$ (solid lines), and $\cosh\beta = 1$ (dashed lines). The amplitude behind a single semi-infinite barrier is shown for comparison (short dashed line).

of $\cosh\beta$ corresponds to the M_2 tide at the entrance to the North Sea (for 2ω we use the local value of the Coriolis parameter f). The appropriate value of $2ak$ in this case (with $2a = 400$ km, $h = 100$ m) is approximately $0{\cdot}29\pi$. The figure also shows the amplitude of the waves behind a single barrier as a function of θ_0 when $\cosh\beta = 2$.

It may be seen that for the North Sea the amplitude differs little from that of a single barrier except for small angles of incidence. When $\theta_0 = \frac{1}{2}\pi$, the amplification of the channel wave is slightly greater than 2. This may be compared with the observed tide in the approaches to the North Sea and the tide at points on the Scottish coast. The average amplitude of the tide at Rockall and in the Shetlands is approximately 2·1 ft., and at points down the Scottish and North-East English coasts it is 4·8 ft.; so the observed amplification equals 2·3 approximately, in fair agreement with the calculated value.

EXAMPLE II. CHANNEL WITH ONE SEMI-INFINITE AND ONE INFINITE BARRIER

We now investigate a model in which long waves approach a semi-infinite barrier in front of an infinite barrier. This and the previous model, together with that of a single semi-infinite barrier (Crease 1956), form three limiting cases for studying the additional effects that may arise at the edges of barriers in rotating systems. Again Heins (1956) has described the results for a similar problem in acoustic diffraction. This corresponds to the special case $\omega = 0$. Much of the detail in this part is the same as for Example I and in general only the outline of the mathematical argument is given.

6. Derivation of the integral equation

Let the barriers be at $y = a$, $-\infty < x < \infty$, and at $y = -a$, $0 < x < \infty$. Again ζ may be expressed (by the use of the boundary condition (5)) as

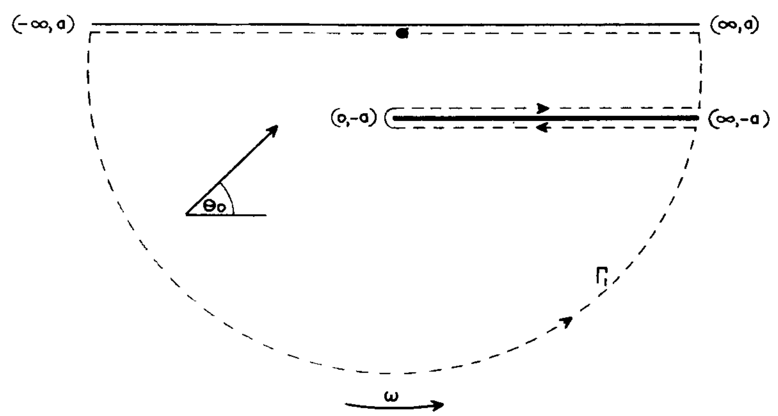

Figure 3.

a contour integral round a contour Γ_1 (figure 3) involving a Green's function to be defined explicitly below; thus

$$\zeta(x,y) = \frac{1}{\cos i\beta}\left[\int_{-\infty}^{\infty}\left\{\zeta(x_0,y_0)\left(\cos i\beta\frac{\partial}{\partial y_0}+\sin i\beta\frac{\partial}{\partial x_0}\right)G_1\right\}_{y_0=a}dx_0+\right.$$
$$\left.+\int_0^{\infty}\left\{[\zeta]\left(\cos i\beta\frac{\partial}{\partial y_0}+\sin i\beta\frac{\partial}{\partial x_0}\right)G_1\right\}_{y=-a}dx_0\right]+$$
$$+\int_{\text{arc}}\left\{G_1\frac{\partial\zeta}{\partial n_0}-\zeta\frac{\partial G_1}{\partial n_0}\right\}ds_0, \quad (31)$$

where the arc in the last integral is the semi-circle shown in figure 3.

Now let $G_1(x,y\,;\,x_0,y_0)$ be chosen so that

$$\left(\cos i\beta\frac{\partial}{\partial y_0}+\sin i\beta\frac{\partial}{\partial x_0}\right)G_1(x,y\,;\,x_0,y_0)=0 \tag{32}$$

on $y=a$. With this definition it is found that the integral round the arc is the sum of the incident wave ζ_i and the wave ζ_r reflected at $y=a$. If $\zeta_i=\exp ik(x\cos\theta_0+y\sin\theta_0)$, then

$$\zeta_r=\frac{\sin(\theta_0-i\beta)}{\sin(\theta_0+i\beta)}\exp ik\{x\cos\theta_0-(y-2a)\sin\theta_0\}. \tag{33}$$

Thus a plane wave reflected at an infinite barrier in a rotating system suffers a change of phase. The two waves together form a wave of Poincaré type (Proudman 1953, p. 265).

Equation (31) now becomes

$$\zeta(x,y)=\frac{1}{\cos i\beta}\int_0^\infty\left\{[\zeta]\left(\cos i\beta\frac{\partial}{\partial y_0}+\sin i\beta\frac{\partial}{\partial x_0}\right)G_1\right\}_{y=-a}dx_0+$$
$$+\exp ik(x\cos\theta_0+y\sin\theta_0)+$$
$$+\frac{\sin(\theta_0-i\beta)}{\sin(\theta_0+i\beta)}\exp ik\{x\cos\theta_0-(y-2a)\sin\theta_0\}. \tag{34}$$

By operating with $\cos i\beta(\cos i\beta\,\partial/\partial y-\sin i\beta\,\partial/\partial x)$ on equation (34) at $y=-a$, $x>0$, the single integral equation for $[\zeta(x_0,-a)]$ of Wiener–Hopf type is found to be

$$g(x)=q(x)+\left(\cos i\beta\frac{\partial}{\partial y}-\sin i\beta\frac{\partial}{\partial y}\right)\times$$
$$\times\int_{-\infty}^{\infty}f(x_0)\left(\cos i\beta\frac{\partial}{\partial y_0}+\sin i\beta\frac{\partial}{\partial x_0}\right)G_1\Big]_{y=y_0=-a}dx_0, \tag{35}$$

where

$$\left.\begin{aligned}
f(x)&=0 && x<0,\\
&=[\zeta(x_0,-a)] && x>0,\\
g(x)&=0 && x>0,\\
q(x)&=0 && x<0,\\
&=2k\cos i\beta\sin(\theta_0-i\beta)\sin(2ak\sin\theta_0)\times &&\\
&\qquad\times\exp ik(x\cos\theta_0+2a\sin\theta_0) && x>0,
\end{aligned}\right\} \tag{36}$$

and $g(x)$ is defined by equation (35) for $x<0$.

7. The Green's function $G_1(x,y;\,x_0,y_0)$

The Green's function satisfying (32) can be derived from the free space Green's function of (6) by expressing it as a contour integral (cf. Watson 1944, p. 178) of the form

$$\tfrac{1}{4}iH_0^{(1)}\{k[(x-x_0)^2+(y-y_0)^2]^{1/2}\}=-\frac{i}{4\pi}\int_{-i\infty+\frac{1}{2}\pi+\psi}^{i\infty-\frac{1}{2}\pi+\psi}\times$$
$$\times\exp ik[(x-x_0)\cos u+(y-y_0)\sin u]\,du, \tag{37}$$

where $x-x_0=R\cos\psi$, $y-y_0=R\sin\psi$. The contour of integration C may be shifted by $\pm\pi/2$ in the direction of the real axis.

It is then clear that G_1 may be represented by

$$G_1(x,y;x_0,y_0) = -\frac{i}{4\pi}\int_C\Big\{\exp ik[(x-x_0)\cos u+(y-y_0)\sin u]+ + \frac{\sin(u+i\beta)}{\sin(u-i\beta)}\exp ik[(x-x_0)\cos u+(y+y_0-2a)\sin u]\Big\}\,du. \tag{38}$$

The second term in the integral represents a source at the image point $(x_0, 2a-y_0)$ with a suitable phase shift to satisfy the boundary condition. The Fourier transform of G_1 regular for $|\mathscr{I}\{\alpha\}| < \mathscr{I}\{k\}$ is found to be

$$K_1(\alpha,y,y_0) = \frac{i}{2(k^2-\alpha^2)^{1/2}}\Big\{\exp i(k^2-\alpha^2)^{1/2}|y-y_0| + \frac{\sin(u_0+i\beta)}{\sin(u_0-i\beta)}\times \times \exp i(k^2-\alpha^2)^{1/2}|y+y_0-2a|\Big\}, \tag{39}$$

where $\sin u_0 = -(k^2-\alpha^2)^{1/2}/k$ and that branch of $(k^2-\alpha^2)^{1/2}$ is chosen which approaches k as α approaches zero.

8. Solution of the integral equation

Taking the Fourier transform of (35) yields formally

$$\begin{aligned} G(\alpha) &= Q(\alpha)+F(\alpha)\Big(\cos i\beta\frac{\partial}{\partial y_0}-i\alpha\sin i\beta\Big)\Big(\cos i\beta\frac{\partial}{\partial y}-i\alpha\sin i\beta\Big)\times \\ &\qquad \times K(\alpha,y,y_0)_{y=y_0=-a} \\ &= Q(\alpha)-F(\alpha)L(\alpha) \end{aligned} \tag{40}$$

say, in notation similar to that used for Example I.

The argument now is essentially as for Example I (there is only one equation to solve, however). $L(\alpha)$ is decomposed into factors $L^-(\alpha)/L^+(\alpha)$ with appropriate regions of regularity, and we find that

$$F(\alpha) = Q(\alpha)L^+(k\cos\theta_0)/L^-(\alpha).$$

Specifically,

$$\left.\begin{aligned} Q(\alpha) &= \frac{2k\cos i\beta}{i(\alpha-k\cos\theta_0)}\sin(\theta_0-i\beta)\sin(2ak\sin\theta_0)\exp(ika\sin\theta_0), \\ L(\alpha) &= \frac{(\alpha^2-k^2\cos^2 i\beta)}{(k^2-\alpha^2)^{1/2}}\sin 2a(k^2-\alpha^2)^{1/2}\exp 2ia(k^2-\alpha^2)^{1/2}, \end{aligned}\right\} \tag{41}$$

and

$$\left.\begin{aligned} L^-(\alpha) &= 2a(\alpha-k\cos i\beta)\exp\Big[\frac{2ia}{\pi}(k^2-\alpha^2)^{1/2}\tan^{-1}\Big(\frac{k+\alpha}{k-\alpha}\Big)^{1/2}+\chi_2(\alpha)\Big]\times \\ &\qquad \times\prod_{n=1}^{\infty}\Big[\Big(1-\frac{4a^2k^2}{n^2\pi}\Big)+\frac{2ia\alpha}{n\pi}\Big]\exp\Big(-\frac{2ia\alpha}{n\pi}\Big), \\ \frac{1}{L^+(\alpha)} &= (\alpha+k\cos i\beta)\exp\Big[\frac{2ia}{\pi}(k^2-\alpha^2)^{1/2}\tan^{-1}\Big(\frac{k-\alpha}{k+\alpha}\Big)^{1/2}-\chi_2(\alpha)\Big]\times \\ &\qquad \times\prod_{n=1}^{\infty}\Big[\Big(1-\frac{4a^2k^2}{n^2\pi^2}\Big)-\frac{2ia\alpha}{n\pi}\Big]\exp\Big(\frac{2ia\alpha}{n\pi}\Big), \end{aligned}\right\} \tag{42}$$

where

$$\chi_2(\alpha) = -\frac{2ia\alpha}{\pi}\left(1-\gamma+\log\frac{ak}{i\pi}\right).$$

9. The wave height ζ

Equation (34) may be expressed in terms of known Fourier transforms as

$$\zeta(x,y) = \frac{1}{2\pi\cos i\beta}\times$$
$$\times\int_{-\infty-i\epsilon}^{\infty-i\epsilon}\left[F(\alpha)\left(\cos i\beta\frac{\partial}{\partial y_0}-i\alpha\sin i\beta\right)K(\alpha,y,y_0)\right]_{y_0=-a}d\alpha+$$
$$+\exp ik(x\cos\theta_0+y\sin\theta_0)+$$
$$+\frac{\sin(\theta_0-i\beta)}{\sin(\theta_0+i\beta)}\exp ik(x\cos\theta_0-(y-2a)\sin\theta_0), \quad (43)$$

where $-\mathscr{I}\{k\}<\epsilon<0$ (or $<\mathscr{I}\{k\cos\theta_0\}<0$). The wave motion at a distance from the edge of the semi-infinite barrier is easily found by evaluating the residues of the integral in (43) at its poles.

First consider the pole at $\alpha = k\cos\theta_0$. This will give rise to the 'geometric' terms ζ_g as we have seen in Example I. When these are combined with the incident and reflected waves, we obtain the following results:

(*a*) $x>0,\quad a>y>-a,$

$$\zeta_g = 0;$$

(*b*) $x>0,\quad -a>y>-\infty,$

$$\zeta_g = \exp ik(x\cos\theta_0+y\sin\theta_0)+\frac{\sin(\theta_0-i\beta)}{\sin(\theta_0+i\beta)}\times$$
$$\times\exp ik(x\cos\theta_0-(y+2a)\sin\theta_0);$$

(*c*) $x<0,\quad a>y>-\infty,$

$$\zeta_g = \exp ik(x\cos\theta_0+y\sin\theta_0)+\frac{\sin(\theta_0-i\beta)}{\sin(\theta_0+i\beta)}\times$$
$$\times\exp ik(x\cos\theta_0-(y-2a)\sin\theta_0).$$

If $0<2ak<\pi$, the only other pole leading to undamped waves at a distance from the edge of the barrier is at $\alpha = k\cos i\beta$. Again, when $x<0$, there is no contribution, and the residue is also zero when $x>0$, $-a>y>-\infty$. For $x>0$, $a>y>-a$, the residue ζ_c at this pole is

$$\zeta_c = \frac{2\cos i\beta\sin\theta_0}{\sin(\theta_0+i\beta)}\left[\frac{\sin(2ak\sin\theta_0)}{2ak\sin\theta_0}\right]^{1/2}\Big/\left[\frac{\sin(2ak\sin i\beta)}{2ak\sin i\beta}\right]^{1/2}\times$$
$$\times\exp\frac{2iak}{\pi}\Big[(\cos i\beta-\cos\theta_0)\left(1-\gamma+\log\frac{i\pi}{2ak}\right)-$$
$$-\theta_0\sin\theta_0-(\pi-i\beta)\sin i\beta+\psi_1\Big]\exp ik(x\cos i\beta+(y+a)\sin i\beta), \quad (44)$$

where ψ_1 is the same as in Example I, except that $2a$ replaces a.

10. Discussion of Example II

There are certain interesting differences between the wave amplitudes in the channel found in this and the previous example.

First, as θ_0 approaches zero in (44), the amplitude approaches zero. Mathematically, the explanation lies in the expression for the reflected wave given by (33). This wave suffers a phase change relative to the incident wave, which approaches 180° as θ_0 becomes small. The effects of the incident and reflected waves thus cancel one another. However, the asymptotic form of (44) as θ_0 approaches zero will be a good approximation only at increasing distances down the barrier. Physically speaking, the particles in a plane wave in a rotating system have transverse accelerations which are balanced by the Coriolis force due to motion in the direction of propagation. When θ_0 approaches zero, these transverse accelerations would be in a plane almost perpendicular to the barrier. As there can be no motion through the barrier, these accelerations must be annulled by those from a reflected wave almost 180° out of phase with the incident wave.

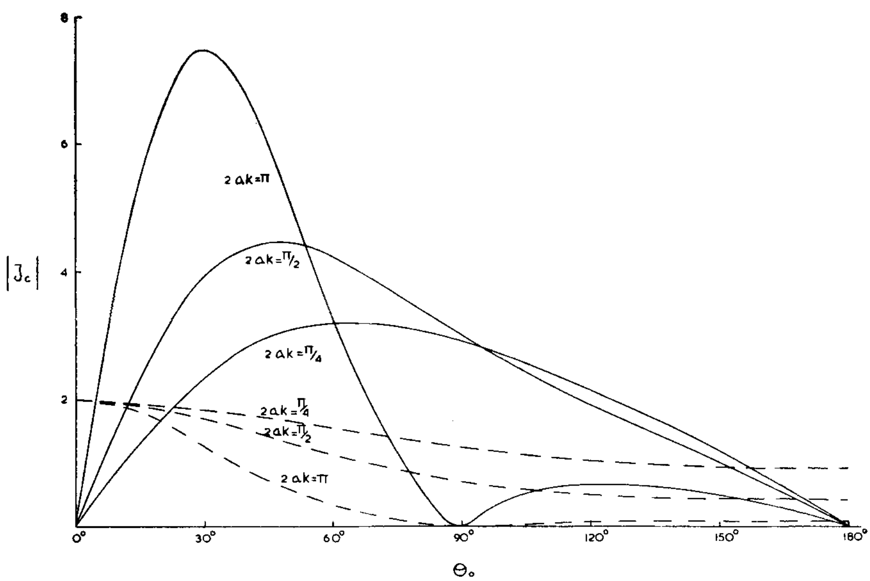

Figure 4. Amplitude of ζ_c on $y = -a$ for cosh $\beta = 2$ (solid lines) and cosh $\beta = 1$ (dashed line).

Second, the term which in Example I was found to arise essentially from the rotational effects at a single semi-infinite barrier (cf. equation (29)) is seen to be absent from equation (44). This again may be understood by considering the rotational effects of two plane waves approaching a single semi-infinite barrier, one at an angle θ_0 and the other at $-\theta_0$ with a phase shift of $\sin(\theta_0 - i\beta)/\sin(\theta_0 + i\beta)$. It is found that the waves propagated down the barrier as a result of these two incident waves annul one another.

Figure 4 gives the amplitude at the barrier $y = -a$, $x > 0$ for $\cosh \beta = 2$ corresponding approximately to the M_2 tide in the latitude of the North Sea. The value of $2ak$ for this sea is $0{\cdot}29\pi$. It is seen that the amplitude of a normally incident wave is about 3·0, which is in excess of the observed value of 2·3 quoted for Example I.

References

Crease, J. 1956 *J. Fluid Mech.* **1**, 86.

Heins, A. E. 1948 *Quart. Appl. Math.* **6**, 157.

Heins, A. E. 1956 *Comm. Pure Appl. Math.* **9**, 447.

Lamb, H. 1932 *Hydrodynamics*, 6th Ed. Cambridge University Press.

Morse, P. M. & Feshbach, H. 1953 *Methods of Theoretical Physics*. New York : McGraw-Hill.

Proudman, J. 1953 *Dynamical Oceanography*. London : Methuen.

Taylor, G. I. 1921 *Proc. Lond. Math. Soc.* (2), **20**, 148.

Titchmarsh, E. C. 1939 *The Theory of Functions*, 2nd Ed. Oxford University Press.

Vajnshtejn, L. A. 1948 *Izv. Akad. Nayk., Ser. Fiz.*, **12**, 144 (translation by J. Shmoys in *Report E.M.–63, Division of Electromagnetic Research, Inst. of Math. Sci., New York University*, 1954).

Watson, G. N. 1944 *Theory of Bessel Functions*, 2nd Ed. Cambridge University Press.